



# Bend-optimal orthogonal drawings of triconnected plane graphs

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## Abstract

A drawing of a plane graph  $G$  in which each edge is represented by a sequence of alternating horizontal and vertical line segments is called an orthogonal drawing. The points of intersection of horizontal and vertical line segments of an edge in an orthogonal drawing are called bends. The best known algorithm to find a bend-optimal orthogonal drawing of a plane graph takes time  $O(n^{1.5})$  where  $n$  is the number of vertices in the graph. In this paper we present a new linear time algorithm to find an orthogonal drawing of a plane 3-connected graph (with maximum degree 4) and give bounds on number of bends (in terms of number  $k$  of degree-4 vertices in the graph) in the resulting drawing with respect to the number  $b(G)$  of bends in the bend-optimal orthogonal drawing of the same graph. The bound is  $b(G) + 3k$ .

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## 1. Introduction

Graph drawings are geometric representations of graphs. They are representations of information and find application in almost every branch of science and technology. The method for laying out data-flow diagrams due to Knuth [1] was one of the first graph drawing algorithms used for visualization purposes. Tamassia [2] has consolidated many graph drawing techniques used in visualization. Graph drawings are also used in VLSI floor planning, VLSI layout, circuit schematics, data flow drawings, RNA genomics etc. Di Battista et al. [3] present a survey on many such algorithms which produce drawings satisfying some aesthetic requirements.

A drawing of a plane graph  $G$  in which each edge is represented by a sequence of alternating horizontal and vertical line segments is called an orthogonal drawing. The points of intersection of horizontal and vertical line segments of an edge in an orthogonal drawing are called bends. A bend-optimal orthogonal drawing of  $G$  is an orthogonal drawing of  $G$  with the minimum number of bends. These drawings are particularly useful in circuit schematics. Fig. 1 shows an

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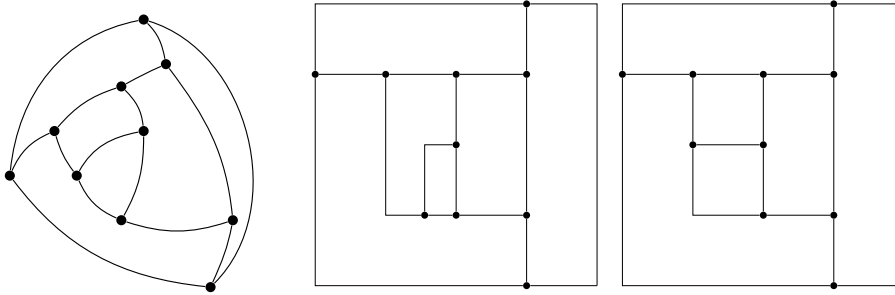


Fig. 1. Orthogonal drawing with 6 and 5 bends of a graph.

example of a plane graph  $G$ , an orthogonal drawing of it and a bend-optimal orthogonal drawing of it. Due to the way they are used in circuit schematics, orthogonal drawings with minimum number of bends are particularly desirable. It is clear from the definition of an orthogonal drawing that a plane graph having a vertex of degree greater than or equal to 5 cannot have an orthogonal drawing. In this paper, we will concentrate on the family of 3-connected graphs with maximum degree 4. Also since the family of plane graphs discussed are 3-connected all the vertices  $v$  have degree either 3 or 4.

Finding a bend-optimal orthogonal drawing of a planar graph  $G$  (with maximum degree 4), is  $NP$ -complete [4]. Bertolazzi et al. [5] present a branch and bound algorithm for computing an orthogonal drawing with minimum number of bends of a biconnected planar graph. Biedl et al. [6,7] study the practical models and 3D orthogonal grid drawings. Di Battista et al. [8] developed a theory on the interplay between spirality and number of bends in orthogonal drawings. For graphs with fixed combinatorial embedding, Tamassia [9] and Garg and Tamassia [10] reduced the bend-optimal orthogonal drawing problem to a minimum cost flow problem and proved that a bend-optimal orthogonal drawing of a given plane graph  $G$  with maximum degree 4 can be computed in  $O(n^2 \log n)$  and  $O(n^{7/4} \sqrt{\log n})$  time, respectively. Over a period of time, much progress has been made and several linear-time algorithms have been presented for finding bend-optimal orthogonal drawings of certain special families of plane graphs. Tamassia and Tollis [11] presented a linear time algorithm for finding an orthogonal drawing for planar graphs. Rahman et al. [12] presented a linear-time algorithm for finding a bend-optimal orthogonal drawing of plane 3-connected cubic graphs. Tamassia et al. [13] give lower bounds on the number of bends along the edges in orthogonal drawings. The best known algorithm for minimizing the number of bends in orthogonal drawings of plane graphs is given by Cornelsen and Karrenbauer [14] in  $O(n^{1.5})$ .

In this paper we give a new algorithm to find an orthogonal drawing of a plane 3-connected graph  $G$  with maximum degree 4. Although this algorithm does not produce a bend-optimal orthogonal drawing of  $G$ , the number of bends is close to optimal if the number of degree 4 vertices is small. Thus it produces an efficient orthogonal drawing of  $G$  both in terms of the number of bends and grid size if the number of degree 4 vertices in  $G$  is small. The basic idea of our algorithm is to transform the given graph  $G$  to a 3-connected cubic graph  $G'$ , find its bend-optimal orthogonal drawing  $D'$  and then construct an orthogonal drawing  $D$  of  $G$  from  $D'$ .

Our main algorithm to find an orthogonal drawing of a plane 3-connected graph  $G$  with maximum degree 4 is presented in Section 3. Section 4 presents bounds on the grid size and the number of bends in the orthogonal drawing obtained by the algorithm. Section 5 gives an open problem we plan to explore.

## 2. Preliminaries

In this section we present some basic definitions and results. We define the *degree* of a vertex as the number of edges incident on it. Throughout this paper let  $G$  be a simple plane 3-connected graph with maximum vertex-degree 4 unless otherwise specified. Let  $V(G)$ ,  $E(G)$  and  $n(G)$  denote vertex set, edge set and cardinality of the vertex set of  $G$ , respectively. A graph is *connected* if each pair of vertices is connected by at least one path. A graph is said to be *k-vertex-connected* (or *k-connected*) if it has at least  $k + 1$  vertices and the graph obtained by deleting any set of less than  $k$  vertices is connected. The *vertex-connectivity* (or *connectivity*) of a graph is the largest integer  $k$  such that it is  $k$ -vertex-connected. A *planar graph* is a graph that can be embedded in the plane (drawn on a plane) in such a way that edges intersect only at their endpoints. A planar graph with a fixed embedding is called a *plane graph*.

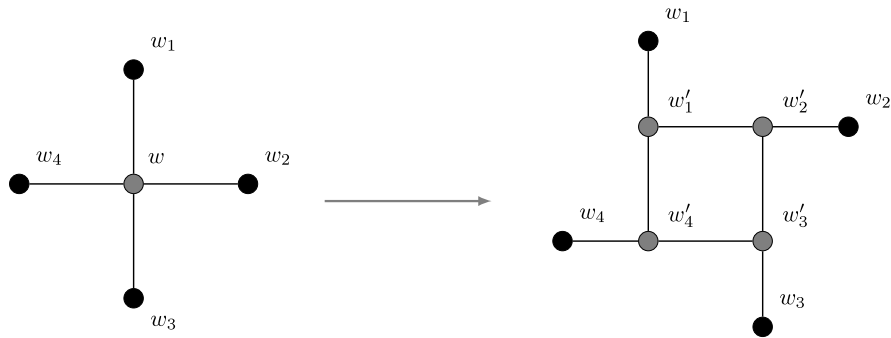


Fig. 2. Replacement of vertex by cycle.

For a simple cycle  $C$  in a plane graph  $G$ , we denote by  $G(C)$  the subgraph of  $G$  inside  $C$  (including  $C$ ). Now we define a new graph operation as follows: Let  $w$  be a vertex of degree  $k$  in a plane graph. Let  $e_1 = ww_1, e_2 = ww_2 \dots e_k = ww_k$  be the edges incident to a vertex  $w$  and assume that these edges appear clockwise around  $w$ . Replace  $w$  with cycle  $w'_1, w'_2 \dots w'_k, w'_1$  and replace the edge  $e_i = ww_i$  by edge  $w'_i w_i$  for  $i = 1, 2 \dots k$  as shown in Fig. 2. We call this operation replacement of vertex by cycle. We will be using this later in our algorithm.

Now we present some known results which will be used later in the paper.

**Lemma 2.1.** *A graph with at least two vertices is  $k$ -connected if and only if for each pair of vertices there exist at least  $k$  vertex-disjoint paths connecting these vertices.*

**Lemma 2.2.** *Let  $G$  be a plane 3-connected cubic graph. Then an orthogonal drawing of  $G$  with minimum number of bends can be found in linear time. [12]*

Our orthogonal drawing algorithm uses the above mentioned linear time algorithm. Rahman, Nakano and Nishizeki have also proved the following result on the grid size of the orthogonal grid drawing of a plane 3-connected cubic graph  $G$  with minimum number of bends:

**Lemma 2.3.** *Let  $G$  be a 3-connected cubic plane graph with  $n$  vertices. Any orthogonal drawing of  $G$  with the minimum number  $b(G)$  of bends has a corresponding orthogonal grid drawing on a grid with width  $W$  and height  $H$  such that*

$$W + H \leq b(G) + \frac{n}{2} - 2, W \leq \frac{n}{2}, H \leq \frac{n}{2} \text{ [12].}$$

### 3. Orthogonal draw algorithm

In this section we present a new algorithm to find an orthogonal drawing of a plane 3-connected graph  $G$  with maximum degree 4. The outline of the algorithm is as follows:

Assume that  $G$  has at least one vertex of degree 4 otherwise it is a plane 3-connected cubic graph and we can find its bend-optimal orthogonal drawing by using algorithm Minimum-Bend of Lemma 2.3 in linear time. We construct a new plane 3-connected cubic graph  $G'$  from  $G$  by performing the replacement of vertex by cycle operation on all vertices of degree 4 in  $G$ . We then find a bend-optimal orthogonal drawing  $D'$  of  $G'$  by using algorithm Minimum-Bend.

We claim the following:

**Lemma 3.1.** *Let  $G'$  be the graph obtained by performing the replacement of vertex by cycle operation on all vertices of degree 4 in a plane 3-connected graph  $G$  with maximum degree 4. Then  $G'$  is a plane 3-connected cubic graph.*

**Proof.** Assume that  $G$  is 3-connected,  $v$  is a vertex of degree 4, and we replace it by a 4-cycle  $C=abcd$  of degree-3 vertices. Let  $G'$  be the resulting graph. Clearly a path in  $G$  traversing  $v$  can be transformed into a path in  $G'$  traversing  $C$ , thus any pair of vertices of  $G' - C$  is connected by three disjoint paths in  $G$ . For two vertices in  $C$ , we can find

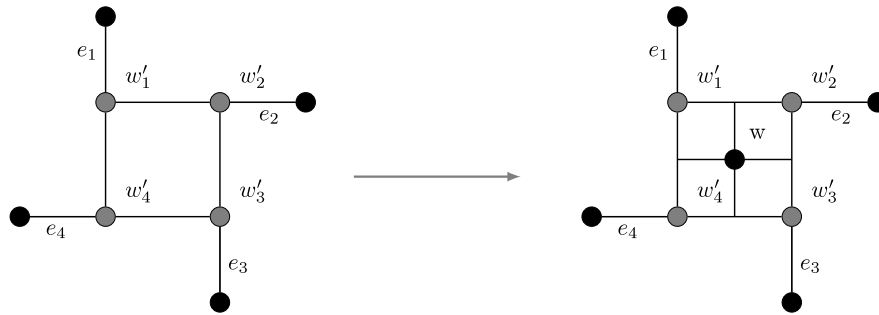


Fig. 3. Dividing rectangular face at mid points of length and width.

two paths in  $C$  connecting them and since  $G' - C = G - v$  is connected, a third path can be found outside  $C$ . Thus it only remains to prove that we can find three disjoint paths between (without loss of generality)  $a$  and some vertex  $u$  not in  $C$ . We start with three vertex disjoint paths between  $u$  and  $v$  in  $G$ . We can assume that one of these paths uses the edge  $e$  incident to  $v$  corresponding to the edge incident to  $a$  that does not belong to  $C$ . If this is not the case, we can simply find a (shortest) path from there to one of the three disjoint paths and then replace that path so it actually uses this edge. Once this is achieved, the three paths can be easily transformed into three paths in  $G'$  from  $a$  to  $u$ . ■

Now by Lemma 2.3 we can find a bend-optimal orthogonal drawing  $D'$  of  $G'$  in linear time. We construct an orthogonal drawing  $D$  of  $G$  from  $D'$  by replacing all the faces in  $D'$  corresponding to degree-4 vertices in  $G$  by a degree-4 vertex by algorithm Orthogonal-Replace such that the drawing retains its orthogonal characteristics. Now since drawing  $D'$  is obtained by the algorithm Minimum-Bend at most one edge in drawing  $D'$  contains two bends and all other edges contain zero or one bend. Since any face in  $D'$  corresponding to degree-4 vertex in  $G$  contains 4 edges, it contains at most 5 bends.

The idea is to find a point inside the face in the grid drawing of  $D'$  corresponding to the degree-4 vertex in  $G$  such that if we draw horizontal and vertical line segments along this point it divides the grid drawing of the face in 4 parts such that none of the parts contain more than 2 points. If the face is rectangular (i.e. none of the edges of the face contain a bend) then we can simply divide at mid points of length and width of the rectangle as shown in Fig. 3.

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**Algorithm 1** Orthogonal Draw

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**Require:** Plane 3-connected graph  $G$  with maximum degree 4

**Ensure:** Orthogonal Drawing of  $G$

- 1: **if**  $\exists v \mid deg(v) = 4$  **then**
  - 2:   Perform replacement of vertex by cycle operation on all degree-4 vertices in  $G$ . Let the resulting graph be  $G'$
  - 3:   Find Orthogonal Drawing  $D'$  of  $G'$  by algorithm Minimum-Bend.
  - 4:   Replace all faces in  $D'$  corresponding to degree-4 vertex in  $G$  by algorithm Orthogonal-Replace to obtain Orthogonal Drawing  $D$  of  $G$
  - 5: **else**
  - 6:   find Orthogonal Drawing  $D$  of  $G$  by algorithm Minimum-Bend.
  - 7: **end if**
  - 8: return Orthogonal Drawing  $D$
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**4. Bound on bends**

In this section we give bounds on number of bends, height and width of the orthogonal drawing obtained by the algorithm Orthogonal-Draw presented in previous section.

**Lemma 4.1.** *Let  $G$  be a plane 3-connected graph with maximum degree 4. Let  $G'$  be the graph obtained by performing the replacement of vertex by cycle operation on all vertices of degree 4 in  $G$ . Let  $b(G)$  and  $b(G')$  be the number of bends in bend-optimal orthogonal drawings of  $G$  and  $G'$ , respectively. Then  $b(G') \leq b(G)$ .*

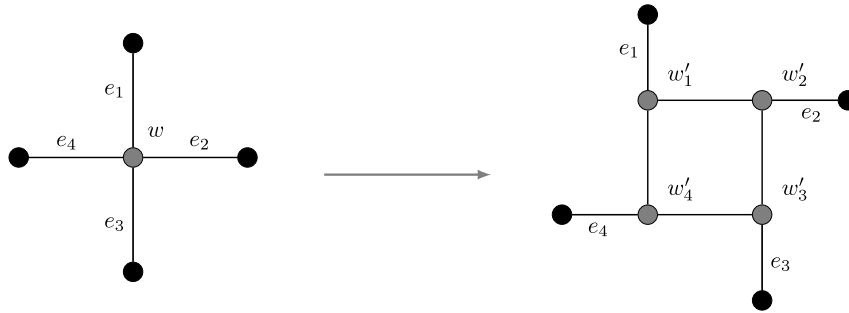


Fig. 4. Replacing degree-4 vertex with a 4-cycle.

**Proof.** Let  $D$  and  $D'$  be bend-optimal orthogonal drawings of  $G$  and  $G'$  with  $b(G)$  and  $b(G')$  bends, respectively. We construct an orthogonal drawing  $D'_{new}$  of  $G'$  from  $D$  by replacing all degree-4 vertices  $w$  in  $D$  with 4-cycles and adjusting the positions of all other vertices in the row or column  $w$  by moving them up, down or sideways according to their position so that the new layout of the drawing still retains orthogonal characteristics as shown in Fig. 4. Since this operation does not add any new bends to the drawing  $D$ , the number of bends in  $D'_{new}$  is equal to  $b(G)$ . Since  $D'$  is a bend-optimal orthogonal drawing of  $G'$ , therefore  $b(G') \leq b(G)$ . ■

We only have to look at the directions of the edges incident to cycle  $C_u$  and use bends to ensure that the edges leave  $u$  with these directions. For example, if all edges leave  $C_u$  in the same direction, then three bends are necessary. This is, however, the maximum possible, since the drawing algorithm from [12] produces at most one bend per edge. In the remaining cases three bends suffice, which yield a total of  $b(G) + 3k$ .

**Lemma 4.2.** Let  $k$  be the number of degree-4 vertices in  $G$ . Then  $H(G) \leq \frac{n(G)+9k-2}{2}$  and  $W(G) \leq \frac{n(G)+9k-2}{2}$  where  $H(G)$  and  $W(G)$  are height and width of orthogonal drawing of  $G$  obtained by the algorithm Orthogonal Draw.

**Proof.** We will prove the bounds for  $H(G)$ , the proof for  $W(G)$  is same. We have  $H(G') + W(G') \leq \frac{n(G')}{2}$ . Now  $n(G') = n(G) + 3k$  and width of drawing  $D'$  of  $G'$  is at least one. Thus  $H(G') \leq \frac{n(G)+3k}{2} - 1$ . Now algorithm Orthogonal-Replace increases the width and height of drawing  $D'$  obtained by Minimum-Bend by at most 3 for each degree-4 vertex. Thus  $H(G) \leq \frac{n(G)+3k}{2} - 1 + 3k = \frac{n(G)+9k-2}{2}$ . ■

## 5. Open problem

It may be possible to modify the algorithm in [12] such that it can compute a bend-minimum orthogonal drawing where every expanded face of  $G'$  has a shape like that in Fig. 2. If this is possible then one immediately gets an algorithm that computes a bend-minimum orthogonal drawing of the input 3-connected plane graph  $G$ , because the bend-stretching transformations would not introduce further bends.

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